DELIGNE'S CONJECTURE ON EXTENSIONS OF 1-MOTIVES

CRISTIANA BERTOLIN

Abstract. We introduce the notion of extension of 1-motives. Using the dictionary between strictly commutative Picard stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, we check that an extension of 1-motives induces an extension of the corresponding strictly commutative Picard stacks. We compute the Hodge, the de Rham and the ℓ-adic realizations of an extention of 1-motives. Using these results we can prove Deligne's conjecture on extensions of 1-motives.

Contents

Introduction

Let k be a field of characteristic 0 embeddable in $\mathbb{C}$. Let $\mathcal{MR}(k)$ be the Tannakian category of mixed realizations (for absolute Hodge cycles) over k. In [D89] Deligne defines the *category of motives* as the Tannakian subcategory of $\mathcal{MR}(k)$ generated by those mixed realizations coming from geometry.

A 1-motive $X = [L \stackrel{u}{\to} E]$ over k is a geometrical object consisting of a finitely generated free $\mathbb{Z}$-module L, an extension E of an abelian variety by a torus, and an homomorphism $u : L \to E$. To each 1-motive X it is possible to associate its Hodge, its ℓ-adic and its De Rham realization. These realizations together with the comparison isomorphisms build a mixed realization $\mathrm{T}(X)$ which is a motive because of the geometrical origin of X.

In [D89] 2.4. Deligne writes: *Je conjecture que l'ensemble des motifs à coefficients entiers de la forme* $\mathrm{T}(X)$*, pour* X *un 1-motif, est stable par extensions. Si* T' *est un motif à coefficients entiers, avec* $\mathrm{T}' \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{T}(X) \otimes \mathbb{Q}$*, alors* T' *est de la forme* $\mathrm{T}(X')$ *avec* X' *isogène à* X*. La conjecture équivaut donc à ce que l'ensemble des motifs* $\mathrm{T}(X) \otimes \mathbb{Q}$*, pour* X *un 1-motif, soit stable par extension. Le mot "conjecture" est abusif en ce que l'énoncé n'a pas un sens précis. Ce qui est*

1991 *Mathematics Subject Classification.* 14C30.
Key words and phrases. Strictly commutative Picard stacks, extensions, 1-motives.

conjecturé est que tout système de réalizations extension de T(X) *par* T(Y) *(X et Y deux 1-motifs), et "naturel", "provenant de la géométrie", est isomorphe à celui défini par un 1-motif Z extension de X par Y.*

In order to explain this conjecture, Deligne furnishes the following example: Let A be an abelian variety over $\mathbb{Q}$. A point a of $A(\mathbb{Q})$ defines a 1-motive $M = [\mathbb{Z} \stackrel{u}{\to} A]$ with $u(1) = a$. The motive $\mathrm{T}(M)$, i.e. the mixed realization defined by M, is an extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(A)$. Therefore we have an arrow

$$\begin{array}{ccc} A(\mathbb{Q}) & \longrightarrow & \mathrm{Ext}^1(\mathrm{T}(\mathbb{Z}), \mathrm{T}(A)) \\ a & \mapsto & \mathrm{T}(M) \end{array}$$

with the Ext^1 computed in the abelian category of motives. Deligne's conjecture applied to $\mathrm{T}(\mathbb{Z})$ and $\mathrm{T}(A)$ says that the above arrow is in fact a bijection:

$$A(\mathbb{Q}) \cong \mathrm{Ext}^1(\mathrm{T}(\mathbb{Z}), \mathrm{T}(A)).$$

In other words, any extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(A)$ in the abelian category of motives (i.e. any mixed realization which is an extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(A)$ and which comes from geometry) is defined by a unique point a of $A(\mathbb{Q})$. The hypothesis *"coming from geometry"* is essential (if we omit it, the conjecture is wrong: see remark 4.1), but present technology gives no way to use it. Therefore, using [By83] (2.2.5), we reformulate Deligne's conjecture on extensions of 1-motives in the following way:

Conjecture 0.1. *Let M_1 and M_2 be two 1-motives defined over a field k of characteristic 0 embeddable in $\mathbb{C}$. There exists a bijection between 1-motives defined over k modulo isogenies which are extensions of M_1 by M_2 and $\mathrm{Ext}^1_{\mathcal{M}(k)}(\mathrm{T}(M_1), \mathrm{T}(M_2))$ in the Tannakian subcategory $\mathcal{M}(k)$ of $\mathcal{MR}(k)$ generated by 1-motives:*

$$\begin{array}{rcl} \varphi : \big\{1 - \text{isomotive } M \text{ extension of } M_1 \text{ by } M_2\big\} & \stackrel{\cong}{\longrightarrow} & \mathrm{Ext}^1_{\mathcal{M}(k)}(\mathrm{T}(M_1), \mathrm{T}(M_2)) \\ M & \mapsto & \mathrm{T}(M). \end{array}$$

Recall that the Tannakian subcategory $\mathcal{M}(k)$ of $\mathcal{MR}(k)$ generated by 1-motives is the strictly full abelian subcategory of $\mathcal{MR}(k)$ which is generated by 1-motives by means of subquotients, direct sums, tensor products and duals. The aim of this paper is to prove the above conjecture.

This paper is organized as followed: in Section 1 we define the notion of extension of 1-motives. In section 2 we recall the notion of extension of strictly commutative Picard stacks, and using the dictionary between strictly commutative Picard stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, we prove that an extension of 1-motives furnishes an extension of the corresponding strictly commutative Picard stacks. In Section 3 we show that there is a bijection between extensions of 1-motives and extensions of the corresponding Hodge realizations. For the ℓ-adic and the De Rham realizations we don't have a bijection but just that extensions of 1-motives define extensions of the corresponding ℓ-adic and De Rham realizations. In Section 4 we prove Conjecture 0.1.

The computation of the group of extensions of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(\mathbb{G}_m)$ in the abelian category of motives

$$\mathbb{G}_m(\mathbb{Q}) \cong \mathrm{Ext}^1(\mathrm{T}(\mathbb{Z}), \mathrm{T}(\mathbb{G}_m))$$

fits into the context of Beilinson's conjectures [Bl87] §5.

In [BK07] Appendix C.9 Barbieri-Viale and Kahn provide a characterisation of the Yoneda Ext in the abelian category of 1-motives with torsion.

Acknowledgment

The author is very grateful to Pierre Deligne who furnishes the good setting in which to work. I want to express my gratitude to Luca Barbieri-Viale for the various discussions we have had on 1-motives.

Notation

Let $\mathbf{S}$ be a site. Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K=(K^i)_i$ such that $K^i=0$ for $i\neq -1$ or 0. The good truncation $\tau_{\leq n}K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leq n}K)^i=K^i$ for $i<n, (\tau_{\leq n}K)^n=\ker(d^n)$ and $(\tau_{\leq n}K)_i=0$ for $i>n$. For any $i\in\mathbb{Z}$, the shift functor $[i]:\mathcal{K}(\mathbf{S})\to\mathcal{K}(\mathbf{S})$ acts on a complex $K=(K^n)_n$ as $(K[i])^n=K^{i+n}$ and $d^n_{K[i]}=(-1)^i d^{n+i}_K$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K)=0$ for $i\neq -1$ or 0. If K and K' are complexes of $\mathcal{D}(\mathbf{S})$, the group $\mathrm{Ext}^i(K,K')$ is by definition $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$ for any $i\in\mathbb{Z}$. Let $\mathrm{RHom}(-,-)$ be the derived functor of the bifunctor $\mathrm{Hom}(-,-)$. The cohomology groups $\mathrm{H}^i\big(\mathrm{RHom}(K,K')\big)$ of $\mathrm{RHom}(K,K')$ are isomorphic to $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$.

1. Extensions of 1-motives

Let S be a scheme.

A *1-motive* $M=(X,A,T,G,u)$ over S consists of

- an S-group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,
- an extension G of an abelian S-scheme A by an S-torus T,
- a morphism $u:X\to G$ of S-group schemes.

A 1-motive $M=(X,A,T,G,u)$ can be viewed also as a complex $[X\stackrel{u}{\to}G]$ of commutative S-group schemes with X concentrated in degree -1 and G concentrated in degree 0. A morphism of 1-motives is a morphism of complexes of commutative S-group schemes. Denote by $1-\mathrm{Mot}(S)$ the category of 1-motives over S. It is an additive category but *it isn't an abelian category.*

Let $S=\mathrm{Spec}\,(k)$ be the spectrum of an algebraically closed field k. Denote by $1-\mathrm{Isomot}(k)$ the $\mathbb{Q}$-linear category associated to the category $1-\mathrm{Mot}(k)$ of 1-motives over k (it has the same objects as $1-\mathrm{Mot}(k)$, but its sets of arrows are the sets of arrows of $1-\mathrm{Mot}(k)$ tensored with $\mathbb{Q}$). The objects of $1-\mathrm{Isomot}(k)$ are called 1-isomotifs and the morphisms of $1-\mathrm{Mot}(k)$ which become isomorphisms in $1-\mathrm{Isomot}(k)$ are the isogenies between 1-motives, i.e. the morphisms of complexes $(f^{-1},f^0):[X\to G]\to[X'\to G']$ such that $f^{-1}:X\to X'$ is injective with finite cokernel and $f^0:G\to G'$ is surjective with finite kernel. *The category* $1-\mathrm{Isomot}(k)$ *is an abelian category.* From now on, we write 1-motive instead of 1-isomotive, unless it is necessary to specify that we work modulo isogenies.

The results of this section are true for any base scheme S such that the category $1-\mathrm{Isomot}(S)$ is abelian. Let $M_1=[X_1\stackrel{u_1}{\to}G_1]$ and $M_2=[X_2\stackrel{u_2}{\to}G_2]$ be two 1-motives defined over such a base scheme S.

Definition 1.1. An *extension* (M,i,j) *of* M_1 *by* M_2 consists of a 1-motive $M = [X \stackrel{u}{\to} G]$ defined over S and two morphisms of 1-motives $i = (i_{-1}, i_0) : M_2 \to M$ and $j = (j_{-1}, j_0) : M \to M_1$

$$\begin{tikzcd} X_2 \arrow[r, hook, "i_{-1}"] \arrow[d, "u_2"'] & X \arrow[r, two heads, "j_{-1}"] \arrow[d, "u"] & X_1 \arrow[d, "u_1"] \\ G_2 \arrow[r, hook, "i_0"] & G \arrow[r, two heads, "j_0"] & G_1 \end{tikzcd} \tag{1.1}$$

such that

- $j_{-1} \circ i_{-1} = 0, j_0 \circ i_0 = 0$,
- i_{-1} and i_0 are injective,
- j_{-1} and j_0 are surjective, and
- u induces an isomorphism between the quotients $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$.

Often we will write only M instead of (M,i,j).

2. Geometrical interpretation

Let $\mathbf{S}$ be a site. A *strictly commutative Picard* $\boldsymbol{S}$*-stack* is an $\mathbf{S}$-stack of groupoids $\mathcal{P}$ endowed with a functor $+ : \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \to \mathcal{P},\ (a,b) \mapsto a+b$, and two natural isomorphisms of associativity σ and of commutativity τ, such that for any object U of $\mathbf{S}$, $(\mathcal{P}(U), +, \sigma, \tau)$ is a strictly commutative Picard category (see [D73] 1.4.2 for more details). An *additive functor* $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$ between strictly commutative Picard $\mathbf{S}$-stacks is a morphism of $\mathbf{S}$-stacks endowed with a natural isomorphism $\sum : F(a+b) \cong F(a) + F(b)$ (for all $a, b \in \mathcal{P}_1$) which is compatible with the natural isomorphisms σ and τ of $\mathcal{P}_1$ and $\mathcal{P}_2$. A *morphism of additive functors* $u : (F, \sum) \to (F', \sum')$ is an $\mathbf{S}$-morphism of $\mathbf{S}$-functors (see [G71] Chapter I 1.1) which is compatible with the natural isomorphisms $\sum$ and $\sum'$.

To any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$ we associate two abelian sheaves: $\pi_0(\mathcal{P})$ the sheaffification of the pre-sheaf which associates to each object U of $\mathbf{S}$ the group of isomorphism classes of objects of $\mathcal{P}(U)$, and $\pi_1(\mathcal{P})$ the sheaf of automorphisms $\underline{\mathrm{Aut}}(e)$ of the neutral object e of $\mathcal{P}$.

Denote by $\mathbf{Picard(S)}$ the category whose objects are small strictly commutative Picard $\mathbf{S}$-stacks and whose arrows are isomorphism classes of additive functors. In [D73] §1.4 Deligne constructs an equivalence of category

$$st : \mathcal{D}^{[-1,0]}(\mathbf{S}) \longrightarrow \mathbf{Picard(S)} \tag{2.1}$$

which furnishes a dictionary between strictly commutative Picard $\mathbf{S}$-stacks and complexes of abelian sheaves on $\mathbf{S}$. We denote by [ ] the inverse equivalence of st.

An *extension* $\mathcal{P} = (\mathcal{P}, I : \mathcal{P}_2 \to \mathcal{P}, J : \mathcal{P} \to \mathcal{P}_1)$ *of* $\mathcal{P}_1$ *by* $\mathcal{P}_2$ consists of a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$, two additive functors $I : \mathcal{P}_2 \to \mathcal{P}$ and $J : \mathcal{P} \to \mathcal{P}_1$, and an isomorphism of additive functors between the composite $J \circ I$ and the trivial additive functor: $J \circ I \cong 0$, such that the following equivalent conditions are satisfied:

- **(a):** $\pi_0(J) : \pi_0(\mathcal{P}) \to \pi_0(\mathcal{P}_1)$ is surjective and I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{P}_2$ and $\ker(J)$;
- **(b):** $\pi_1(I) : \pi_1(\mathcal{P}_2) \to \pi_1(\mathcal{P})$ is injective and J induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathrm{coker}(I)$ and $\mathcal{P}_1$.

(see [Be10] for the definition of kernel and cokernel of an additive functor). Let $K = [K^{-1} \stackrel{d^K}{\to} K^0]$ and $L = [L^{-1} \stackrel{d^L}{\to} L^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, and let $F : st(K) \to st(L)$ be an additive functor induced by a morphism of complexes $f : K \to L$ in $\mathcal{K}^{[-1,0]}(\mathbf{S})$. According to [Be10] Lemma 3.4 we have

$$[\ker(F)] = \tau_{\leq 0}\big(MC(f)[-1]\big) = \big[K^{-1} \stackrel{(f^{-1},-d^K)}{\longrightarrow} \ker(d^L, f^0)\big] \tag{2.2}$$

$$[\mathrm{coker}(F)] = \tau_{\geq -1} MC(f) = \big[\mathrm{coker}(f^{-1}, -d^K) \stackrel{(d^L,f^0)}{\longrightarrow} L^0\big] \tag{2.3}$$

where $MC(f)$ is the mapping cone of the morphism f. Hence we have the following

Corollary 2.1. *Let*

$$K \stackrel{i}{\longrightarrow} L \stackrel{j}{\longrightarrow} M$$

be morphisms of complexes of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$ and denote by I and J the additive functors induced by i and j respectively. Then the strictly commutative Picard $\boldsymbol{S}$-stack $st(L) = (st(L), I, J)$ is an extension of $st(M)$ by $st(K)$ if and only if $j \circ i = 0$ and the following equivalent conditions are satisfied:

- **(a):** $\mathrm{H}^0(j) : \mathrm{H}^0(L) \to \mathrm{H}^0(M)$ *is surjective and i induces a quasi-isomorphism between K and $\tau_{\leq 0}(MC(j)[-1])$;*
- **(b):** $\mathrm{H}^{-1}(i) : \mathrm{H}^{-1}(K) \to \mathrm{H}^{-1}(L)$ *is injective and j induces a quasi-isomorphism between $\tau_{\geq -1}MC(i)$ and M.*

Let S be a scheme. From now on the site $\mathbf{S}$ is the big fppf site over S.

Proposition 2.2. *Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over S. If $M = [X \stackrel{u}{\to} G]$ is an extension of M_1 by M_2, then $st(M)$ is an extension of $st(M_1)$ by $st(M_2)$.*

Proof. Denote by $(i_{-1}, i_0) : M_2 \to M$ and $(j_{-1}, j_0) : M \to M_1$ the morphisms of 1-motives underlying the extension M of M_1 by M_2. These morphisms furnish two additive functors:

$$I : st(M_2) \longrightarrow st(M) \qquad \text{and} \qquad J : st(M) \longrightarrow st(M_1).$$

First observe that the conditions $j_{-1} \circ i_{-1} = 0$ and $j_0 \circ i_0 = 0$ imply that $J \circ I \cong 0$. Remark also that since $j_0 : G \to G_1$ is surjective, also the morphism $\mathrm{H}^0(j) : G/u(X) \to G_1/u_1(X_1)$ is surjective.
By Corollary 2.1, it remains to prove that the morphism of complexes i induces a quasi-isomorphism between M_2 and $\tau_{\leq 0}(MC(j)[-1])$. Explicitly $\tau_{\leq 0}(MC(j)[-1])$ is the complex

$$[X \stackrel{k}{\longrightarrow} \ker(u_1, j_0)]$$

with $k : X \to \ker(u_1, j_0)$ the morphism induced by $(j_{-1}, -u) : X \to X_1 + G$, and so we have to prove that (i_{-1}, i_0) induces the quasi-isomorphisms

$$\ker(u_2) \cong \ker(k), \tag{2.4}$$

$$G_2/u_2(X_2) \cong \ker(u_1, j_0)/(j_{-1}, -u)(X). \tag{2.5}$$

We start with the first isomorphism. Because of the commutativity of the first square of diagram (1.1), $i_{-1}(\ker(u_2))$ is contained in $\ker(u)$. Since $j_{-1} \circ i_{-1} = 0$, $i_{-1}(\ker(u_2))$ is contained also in $\ker(j_{-1})$ and so we have the inclusion $i_{-1}(\ker(u_2)) \subseteq$

$\ker(k)$. The isomorphism between the quotients $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$ induces the exact sequence

$$0 \longrightarrow X_2 \xrightarrow{i_{-1}} \ker(j_{-1}) \xrightarrow{u} \ker(j_0)/\mathrm{im}(i_0) \longrightarrow 0$$

Therefore we have the equality $\ker(k) \subseteq i_{-1}(X_2)$. Now because of the commutativity of the first square of diagram (1.1) and because of the injectivity of i_0 we have that $i_{-1}(\ker(u_2))$ contains $\ker(k)$. Hence we can conclude that via the morphism i_{-1}, $\ker(u_2)$ and $\ker(k)$ are isomorphic.
Concerning the second isomorphism (2.5), since $j_{-1} : X \to X_1$ is surjective, we have the isomorphism $\ker(u_1, j_0)/(j_{-1}, -u)(X) \cong \ker(j_0)/u(X)$, and so we have to prove that the morphism $i_0 : G_2 \to G$ induces an isomorphism

$$G_2/u_2(X_2) \cong \ker(j_0)/u(X).$$

Since the morphism $u : X \to G$ induces the isomorphism $\ker(j_{-1})/i_{-1}(X_2) \cong \ker(j_0)/i_0(G_2)$, the composite of the injection $i_0 : G_2 \to \ker(j_0)$ with the projection $\ker(j_0) \to \ker(j_0)/u(X)$ furnishes the surjection

$$p : G_2 \longrightarrow \ker(j_0)/u(X).$$

Because of the commutativity of the first square of diagram (1.1), $u_2(X_2)$ is contained in $\ker(p)$. On the other hand $i_0(\ker(p))$ is contained in $u(X)$. The isomorphism $\ker(j_{-1})/i_{-1}(X_2) \cong \ker(j_0)/i_0(G_2)$ implies that in fact $i_0(\ker(p))$ is contained in $u(i_{-1}(X_2))$. Because of the commutativity of the first square of diagram (1.1) and because of the injectivity of i_0, $\ker(p)$ is contained in $u_2(X_2)$. Hence via the morphism i_0, $G_2/u_2(X_2)$ and $\ker(j_0)/u(X)$ are isomorphic. $\square$

By the above proposition, the group law for extensions of strictly commutative Picard **S**-stacks defined in [Be10] §4 furnishes a group law for extensions of 1-motives. The neutral object with respect to this group law on the set of isomorphism classes of extensions of $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ by $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ is the 1-motive $M_1 + M_2 = [X_1 \times X_2 \stackrel{(u_1,u_2)}{\to} G_1 \times G_2]$.

3. Transcendental and algebraic interpretations

First we recall briefly the construction of the Hodge, De Rham and ℓ-adic realizations of a 1-motive $M = (X, A, T, G, u)$ defined over S (see [D74] §10.1 for more details):

- if S is the spectrum of the field $\mathbb{C}$ of complex numbers, the *Hodge realization* $\mathrm{T_H}(M) = (\mathrm{T}_{\mathbb{Z}}(M), \mathrm{W}_*, \mathrm{F}^*)$ of M is the mixed Hodge structure consisting of the fibred product $\mathrm{T}_{\mathbb{Z}}(M) = \mathrm{Lie}(G) \times_G X$ (viewing $\mathrm{Lie}(G)$ over G via the exponential map and X over G via u) and of the weight and Hodge filtrations defined in the following way:

$$\begin{aligned}
\mathrm{W}_0(\mathrm{T}_{\mathbb{Z}}(M)) &= \mathrm{T}_{\mathbb{Z}}(M),\\
\mathrm{W}_{-1}(\mathrm{T}_{\mathbb{Z}}(M)) &= \mathrm{H}_1(G, \mathbb{Z}),\\
\mathrm{W}_{-2}(\mathrm{T}_{\mathbb{Z}}(M)) &= \mathrm{H}_1(T, \mathbb{Z}),\\
\mathrm{F}^0(\mathrm{T}_{\mathbb{Z}}(M) \otimes \mathbb{C}) &= \ker\big(\mathrm{T}_{\mathbb{Z}}(M) \otimes \mathbb{C} \longrightarrow \mathrm{Lie}(G)\big).
\end{aligned}$$

- if S is the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$, the *ℓ-adic realization* $\mathrm{T}_\ell(M)$ of the 1-motive M is the projective limit of the $\mathbb{Z}/\ell^n\mathbb{Z}$-modules

$$\mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}}(M) = \big\{(x,g) \in X \times G \mid u(x) = \ell^n\, g\big\} \big/ \big\{(\ell^n\, x, u(x)) \mid x \in X\big\}.$$

- if S is the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$, the *de Rham realization* $\mathrm{T}_{\mathrm{dR}}(M)$ of M is the Lie algebra of $G^\natural$ where $M^\natural = [X \to G^\natural]$ is the universal vectorial extension of M by the vectorial group $\mathrm{Ext}^1(M, \mathbb{G}_a)^*$. The Hodge filtration on $\mathrm{T}_{\mathrm{dR}}(M)$ is defined by $\mathrm{F}^0\mathrm{T}_{\mathrm{dR}}(M) = \ker(\mathrm{Lie}\, G^\natural \to \mathrm{Lie}\, G)$.

Proposition 3.1. *Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over $\mathbb{C}$.*

(1) *If $M = [X \stackrel{u}{\to} G]$ is an extension of M_1 by M_2, then $\mathrm{T}_{\mathrm{H}}(M)$ is an extension of $\mathrm{T}_{\mathrm{H}}(M_1)$ by $\mathrm{T}_{\mathrm{H}}(M_2)$ in the abelian category $\mathcal{MHS}$ of mixed Hodge structures.*
(2) *Let E be an extension of $\mathrm{T}_{\mathrm{H}}(M_1)$ by $\mathrm{T}_{\mathrm{H}}(M_2)$ in the category $\mathcal{MHS}$. Then modulo isogenies, there exists a unique extension M of M_1 by M_2 which defines the isomorphism class of the extension E i.e. such that $\mathrm{T}_{\mathrm{H}}(M)$ and E are isomorphic in $\mathcal{MHS}$ as extensions of $\mathrm{T}_{\mathrm{H}}(M_1)$ by $\mathrm{T}_{\mathrm{H}}(M_2)$.*

In other words, we have a bijection

$$\begin{array}{rcl} \varphi : \big\{1 - \text{isomotive } M \text{ extension of } M_1 \text{ by } M_2\big\} & \stackrel{\cong}{\longrightarrow} & \mathrm{Ext}^1_{\mathcal{MHS}}(\mathrm{T}_{\mathrm{H}}(M_1), \mathrm{T}_{\mathrm{H}}(M_2)) \\ M & \mapsto & \mathrm{T}_{\mathrm{H}}(M). \end{array}$$

Proof. (1) Denote by $i = (i_{-1}, i_0) : M_2 \to M$ and $j = (j_{-1}, j_0) : M \to M_1$ the morphisms of 1-motives underlying the extension $M = (M, i, j)$. By Proposition 2.2, the strictly commutative Picard $\mathbf{S}$-stack $st(M)$ is an extension of $st(M_1)$ by $st(M_2)$. Corollary 2.1 implies that via i the complexes M_2 and $\tau_{\leq 0}(MC(j)[-1])$ are isomorphic in the derived category $\mathcal{D}(\mathbf{S})$, and so, via the morphism $\mathrm{T}_{\mathrm{H}}(i_{-1}, i_0)$ induced by $i = (i_{-1}, i_0)$, their Hodge realizations are isomorphic in the category $\mathcal{MHS}$:

$$\mathrm{T}_{\mathrm{H}}(i_{-1}, i_0) : \mathrm{T}_{\mathrm{H}}(M_2) \stackrel{\cong}{\longrightarrow} \mathrm{T}_{\mathrm{H}}(\tau_{\leq 0}(MC(j)[-1])).$$

Explicitly the $\mathbb{Z}$-module underlying the Hodge realization of $\tau_{\leq 0}(MC(j)[-1])$ is

$$\begin{array}{rcl} \mathrm{T}_{\mathbb{Z}}(\tau_{\leq 0}(MC(j)[-1])) & = & \mathrm{Lie}\,(\ker(u_1, j_0)) \times_{\ker(u_1, j_0)} X \\ & = & \big(\mathrm{Lie}\,(\ker(j_0)) \oplus \ker(u_1)\big) \times_{\ker(u_1, j_0)} X \end{array} \tag{3.1}$$

The morphism of 1-motive $j = (j_{-1}, j_0) : M \to M_1$ induces a morphism $\mathrm{T}_{\mathrm{H}}(j_{-1}, j_0) : \mathrm{T}_{\mathrm{H}}(M) \to \mathrm{T}_{\mathrm{H}}(M_1)$ between the Hodge realizations of M and M_1. To have this morphism is the same as to have the morphisms $\mathrm{Lie}\,(j_0) : \mathrm{Lie}\,(G) \to \mathrm{Lie}\,(G_1)$ and

$j_{-1} : X \to X_1$ such that the following diagram commute

(3.2)
$$\begin{tikzcd}
 & \mathrm{Lie}\,(G) \arrow[rr, "\mathrm{Lie}\,(j_0)"] & & \mathrm{Lie}\,(G_1) \arrow[dr, "exp"] & \\
\mathrm{Lie}\,(G) \times_G X \arrow[ur, "pr"] \arrow[rr, "\mathrm{T}_{\mathbb{Z}}(j_{-1},j_0)"] \arrow[dr, "pr"'] & & \mathrm{Lie}\,(G_1) \times_{G_1} X_1 \arrow[ur, "pr"] \arrow[dr, "pr"'] & & G_1 \\
 & X \arrow[rr, "j_{-1}"'] & & X_1 \arrow[ur, "u_1"'] &
\end{tikzcd}$$

where pr are the projections and exp the exponential map. Since the morphisms $j_{-1} : X \to X_1$ and $j_0 : G \to G_1$ are surjective, also the morphism $\mathrm{T_H}(j_{-1}, j_0)$ is surjective. Moreover the equality (3.1) implies that the mixed Hodge structure $\mathrm{T_H}(\tau_{\leq 0}(MC(j)[-1]))$ is the kernel of $\mathrm{T_H}(j_{-1}, j_0) : \mathrm{T_H}(M) \to \mathrm{T_H}(M_1)$. Hence we have an exact sequence in the category $\mathcal{MHS}$

$$0 \longrightarrow \mathrm{T_H}(M_2) \overset{\mathrm{T_H}(i_{-1},i_0)}{\longrightarrow} \mathrm{T_H}(M) \overset{\mathrm{T_H}(j_{-1},j_0)}{\longrightarrow} \mathrm{T_H}(M_1) \longrightarrow 0.$$

Setting $\varphi(M) = \mathrm{T_H}(M)$ we have construct an arrow

$$\varphi : \big\{1-\text{isomotive } M \text{ extension of } M_1 \text{ by } M_2\big\} \longrightarrow \mathrm{Ext}^1_{\mathcal{MHS}}(\mathrm{T_H}(M_1), \mathrm{T_H}(M_2))$$

which is well defined: isogeneous 1-motives which are extensions of M_1 by M_2 define the same isomorphism class of extensions of $\mathrm{T_H}(M_1)$ by $\mathrm{T_H}(M_2)$. The reader can check that the arrow φ is in fact an homomorphism, i.e. it respects the group law of extensions of 1-motives and the group law of extensions of mixed Hodge structures.
(2) Now we prove that φ is a bijection.
Injectivity of φ : Let M be a 1-motive extension of M_1 by M_2 and suppose that $\varphi(M)$ is the zero object of $\mathrm{Ext}^1_{\mathcal{MHS}}(\mathrm{T_H}(M_1), \mathrm{T_H}(M_2))$. We have

$$\begin{aligned} \mathrm{T_H}(M) &= \mathrm{T_H}(M_1) \oplus \mathrm{T_H}(M_2), \\ &= \mathrm{Lie}\,(G_1 \times G_2) \times_{G_1 \times G_2} (X_1 \times X_2). \end{aligned}$$

Therefore the 1-motives M and $[X_1 \times X_2 \overset{u_1 \times u_2}{\to} G_1 \times G_2]$ have the same Hodge realization and so they are isogeneous.
Surjectivity of φ : Now suppose to have an extension E of $\mathrm{T_H}(M_1)$ by $\mathrm{T_H}(M_2)$ in the category $\mathcal{MHS}$

$$0 \longrightarrow \mathrm{T_H}(M_2) \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} \mathrm{T_H}(M_1) \longrightarrow 0.$$

Since $\mathrm{T_H}(M_1)$ and $\mathrm{T_H}(M_2)$ are mixed Hodge structures of type $\{0,0\}, \{-1,0\}$, $\{0,-1\}, \{-1,-1\}$ also E must be of this type. Therefore according to the equivalence of category [D74] (10.1.3), there exists a 1-motive M and morphisms of 1-motives $i = (i_{-1}, i_0) : M_2 \to M, j = (j_{-1}, j_0) : M \to M_1$ such that $\mathrm{T_H}(M) = E$ and $\mathrm{T_H}(i) = f, \mathrm{T_H}(j) = g$. It remains to check that (M, i, j) is an extension of M_1 by M_2. Since $g \circ f = 0$, it is clear that $j \circ i = 0$. Because of the commutative diagram (3.2), the surjectivity of g implies the surjectivity of $j_0 : G \to G_1$ and of $j_{-1} : X \to X_1$. Doing an analogous commutative diagram for the morphism $f = \mathrm{T_H}(i) : \mathrm{Lie}\,(G_2) \times_{G_2} X_2 \to \mathrm{Lie}\,(G) \times_G X$, we see that the injectivity of f implies the injectivity of $i_0 : G_2 \to G$ and of $i_{-1} : X_2 \to X$. Let now m be an element of $\mathrm{T_H}(M) = \mathrm{Lie}\,(G) \times_G X$. We have that $\mathrm{T_H}(j)(m) = 0$ if the projection $pr_{\mathrm{Lie}\,(G)}(m)$ of m on $\mathrm{Lie}\,(G)$ lies in $\ker(\mathrm{Lie}\,(j_0))$, and the projection $pr_X(m)$ of m on

X lies in $\ker(j_{-1})$. Hence the morphism $u : X \to G$ has to induce an isomorphism between $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$. □

Proposition 3.2. *Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over a field k of characteristic 0 embeddable in $\mathbb{C}$. If $M = [X \stackrel{u}{\to} G]$ is an extension of M_1 by M_2, then $\mathrm{T}_\ell(M)$ is an extension of $\mathrm{T}_\ell(M_1)$ by $\mathrm{T}_\ell(M_2)$.*

Proof. Denote by $i = (i_{-1}, i_0) : M_2 \to M$ and $j = (j_{-1}, j_0) : M \to M_1$ the morphisms of 1-motives underlying the extension $M = (M, i, j)$. By Proposition 2.2, the strictly commutative Picard $\mathbf{S}$-stack $st(M)$ is an extension of $st(M_1)$ by $st(M_2)$. Corollary 2.1 implies that via i the complexes M_2 and $\tau_{\leq 0}(MC(j)[-1])$ are isomorphic in the derived category $\mathcal{D}(\mathbf{S})$, and so, via the morphism $\mathrm{T}_\ell(i_{-1}, i_0)$ induced by $i = (i_{-1}, i_0)$, their ℓ-adic realizations are isomorphic:

$$\mathrm{T}_\ell(i_{-1}, i_0) : \mathrm{T}_\ell(M_2) \xrightarrow{\cong} \mathrm{T}_\ell(\tau_{\leq 0}(MC(j)[-1])).$$

Explicitly the ℓ-adic realization of $\tau_{\leq 0}(MC(j)[-1])$ is the projective limit of the $\mathbb{Z}/\ell^n\mathbb{Z}$-modules

$$\mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}}(\tau_{\leq 0}(MC(j)[-1])) = \tag{3.3}$$

$$\big\{(x,(z,g)) \in X \times \ker(u_1, j_0) \mid (j_{-1}, -u)(x) = \ell^n\,(z,g)\big\} \big/ \big\{(\ell^n\,x, (j_{-1}, -u)(x)) \mid x \in X\big\}$$

The morphism of 1-motive $j = (j_{-1}, j_0) : M \to M_1$ induces a morphism $\mathrm{T}_\ell(j_{-1}, j_0) : \mathrm{T}_\ell(M) \to \mathrm{T}_\ell(M_1)$ between the ℓ-adic realizations of M and M_1. Since the morphisms $j_{-1} : X \to X_1$ and $j_0 : G \to G_1$ are surjective, also the morphism $\mathrm{T}_\ell(j_{-1}, j_0)$ is surjective. Moreover from the equality (3.3) we get that the $\mathbb{Q}_\ell$-vector space $\mathrm{T}_\ell(\tau_{\leq 0}(MC(j)[-1]))$ is the kernel of the morphism $\mathrm{T}_\ell(j_{-1}, j_0) : \mathrm{T}_\ell(M) \to \mathrm{T}_\ell(M_1)$. Hence we have an exact sequence

$$0 \longrightarrow \mathrm{T}_\ell(M_2) \stackrel{\mathrm{T}_\ell(i_{-1},i_0)}{\longrightarrow} \mathrm{T}_\ell(M) \stackrel{\mathrm{T}_\ell(j_{-1},j_0)}{\longrightarrow} \mathrm{T}_\ell(M_1) \longrightarrow 0.$$

□

Proposition 3.3. *Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over a field k of characteristic 0 embeddable in $\mathbb{C}$. If $M = [X \stackrel{u}{\to} G]$ is an extension of M_1 by M_2, then $\mathrm{T}_{\mathrm{dR}}(M)$ is an extension of $\mathrm{T}_{\mathrm{dR}}(M_1)$ by $\mathrm{T}_{\mathrm{dR}}(M_2)$.*

Proof. Denote by $i = (i_{-1}, i_0) : M_2 \to M$ and $j = (j_{-1}, j_0) : M \to M_1$ the morphisms of 1-motives underlying the extension $M = (M, i, j)$. By Proposition 2.2, the strictly commutative Picard $\mathbf{S}$-stack $st(M)$ is an extension of $st(M_1)$ by $st(M_2)$. Corollary 2.1 implies that via i the complexes M_2 and $\tau_{\leq 0}(MC(j)[-1])$ are isomorphic in the derived category $\mathcal{D}(\mathbf{S})$, and so, via the morphism $\mathrm{T}_{\mathrm{dR}}(i_{-1}, i_0)$ induced by $i = (i_{-1}, i_0)$, their de Rham realizations are isomorphic:

$$\mathrm{T}_{\mathrm{dR}}(i_{-1}, i_0) : \mathrm{T}_{\mathrm{dR}}(M_2) \xrightarrow{\cong} \mathrm{T}_{\mathrm{dR}}(\tau_{\leq 0}(MC(j)[-1])).$$

Explicitly the de Rham realization of the 1-motive $\tau_{\leq 0}(MC(j)[-1])$ is

$$\begin{aligned} \mathrm{T}_{\mathrm{dR}}(\tau_{\leq 0}(MC(j)[-1])) &= \mathrm{Lie}\left(\ker(u_1, j_0)^\natural\right) \\ &= \mathrm{Lie}\left(\ker(j_0)^\natural\right) \oplus \left(\ker(u_1) \otimes k\right) \end{aligned} \tag{3.4}$$

where $(\tau_{\leq 0}(MC(j)[-1]))^\natural = [X \to \ker(u_1, j_0)^\natural]$ is the universal vectorial extension of $\tau_{\leq 0}(MC(j)[-1])$ by the vectorial group $\mathrm{Ext}^1(\tau_{\leq 0}(MC(j)[-1]), \mathbb{G}_a)^*$. The morphism of 1-motive $j = (j_{-1}, j_0) : M \to M_1$ induces a morphism $\mathrm{T}_{\mathrm{dR}}(j_{-1}, j_0)$:

$\mathrm{T}_{\mathrm{dR}}(M) \to \mathrm{T}_{\mathrm{dR}}(M_1)$ between the de Rham realizations of M and M_1. Explicitly we have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{T}_{\mathrm{dR}}(M) = \mathrm{Lie}\,(G^\natural) \arrow[dr, "exp"] & & & X \arrow[dll] \arrow[r, equal] & X \arrow[dll, "u"'] \arrow[ddd, "j_{-1}"] \\
0 \arrow[r] & \mathrm{Ext}^1(\tau_{\leq 0}(M, \mathbb{G}_a))^* \arrow[r] \arrow[d] & G^\natural \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "j_0"] & 0 & \\
0 \arrow[r] & \mathrm{Ext}^1(\tau_{\leq 0}(M_1, \mathbb{G}_a))^* \arrow[r] & G_1^\natural \arrow[r] & G_1 \arrow[r] & 0 & \\
 & \mathrm{T}_{\mathrm{dR}}(M_1) = \mathrm{Lie}\,(G_1^\natural) \arrow[ur, "exp"'] & & & X_1 \arrow[ull] \arrow[r, equal] & X_1 \arrow[ull, "u_1"]
\end{tikzcd}$$

where *exp* are the exponential maps. Since the morphisms $j_{-1} : X \to X_1$ and $j_0 : G \to G_1$ are surjective, also the morphism $\mathrm{T}_{\mathrm{dR}}(j_{-1}, j_0)$ is surjective. Moreover the equality (3.4) implies that the k-vector space $\mathrm{T}_{\mathrm{dR}}(\tau_{\leq 0}(MC(j)[-1]))$ is the kernel of $\mathrm{T}_{\mathrm{dR}}(j_{-1}, j_0) : \mathrm{T}_{\mathrm{dR}}(M) \to \mathrm{T}_{\mathrm{dR}}(M_1)$. Hence we have an exact sequence

$$0 \longrightarrow \mathrm{T}_{\mathrm{dR}}(M_2) \stackrel{\mathrm{T}_{\mathrm{dR}}(i_{-1},i_0)}{\longrightarrow} \mathrm{T}_{\mathrm{dR}}(M) \stackrel{\mathrm{T}_{\mathrm{dR}}(j_{-1},j_0)}{\longrightarrow} \mathrm{T}_{\mathrm{dR}}(M_1) \longrightarrow 0.$$

$\square$

4. Proof of the conjecture

Let S be the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$. Fix an algebraic closure $\overline{k}$ of k. Let $\mathcal{MR}(k)$ be the neutral Tannakian category over $\mathbb{Q}$ of mixed realizations (for absolute Hodge cycles) over k. The objects of $\mathcal{MR}(k)$ are families

$$N = ((N_\sigma, \mathcal{L}_\sigma), N_{\mathrm{dR}}, N_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\ell,\sigma,\overline{\sigma}}$$

where

- N_σ is a mixed Hodge structure for any embedding $\sigma : k \to \mathbb{C}$ of k in $\mathbb{C}$;
- N_{dR} is a finite dimensional k-vector space with an increasing filtration W_* (the Weight filtration) and a decreasing filtration F^* (the Hodge filtration);
- N_ℓ is a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous $\mathrm{Gal}(\overline{k}/k)$-action and an increasing filtration W_* (the Weight filtration), which is $\mathrm{Gal}(\overline{k}/k)$-equivariant, for any prime number ℓ;
- $I_{\sigma,\mathrm{dR}} : N_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to N_{\mathrm{dR}} \otimes_k \mathbb{C}$ and $I_{\overline{\sigma},\ell} : N_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to N_\ell$ are comparison isomorphisms for any ℓ, any σ and any $\overline{\sigma}$ extension of σ to the algebraic closure of k;
- $\mathcal{L}_\sigma$ is a lattice in N_σ such that, for any prime number ℓ, the image $\mathcal{L}_\sigma \otimes \mathbb{Z}_\ell$ of this lattice through the comparison isomorphism $I_{\overline{\sigma},\ell}$ is a $\mathrm{Gal}(\overline{k}/k)$-invariant subgroup of N_ℓ ($\mathcal{L}_\sigma$ is the integral structure of the object N of $\mathcal{MR}_{\mathbb{Z}}(k)$).

According to [D74] (10.1.3) we have the fully faithful functor

$$\begin{array}{rcl} 1 - \mathrm{Mot}(k) & \longrightarrow & \mathcal{MR}(k) \\ M & \longmapsto & \mathrm{T}(M) = (\mathrm{T}_\sigma(M), \mathrm{T}_{\mathrm{dR}}(M), \mathrm{T}_\ell(M), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\ell,\sigma,\overline{\sigma}} \end{array}$$

which attaches to each 1-motive M its Hodge realization $\mathrm{T}_\sigma(M)$ for any embedding $\sigma : k \to \mathbb{C}$ of k in $\mathbb{C}$, its de Rham realization $\mathrm{T}_{\mathrm{dR}}(M)$, its ℓ-adic realization

$\mathrm{T}_\ell(M)$ for any prime number ℓ, and its comparison isomorphisms. Denote by $\mathcal{M}(k)$ the Tannakian subcategory of $\mathcal{MR}(k)$ generated by 1-motives, i.e. the strictly full abelian subcategory of $\mathcal{MR}(k)$ which is generated by 1-motives by means of subquotients, direct sums, tensor products and duals. Recall that according to [By83] (2.2.5), any embedding $\sigma : k \to \mathbb{C}$ of k in $\mathbb{C}$ furnishes a fully faithful functor from $\mathcal{M}(k)$ to the category $\mathcal{MHS}$ of mixed Hodge structures.

We can now prove Conjecture 0.1:

Proof. Denote by $\mathrm{T}(M_i) = (\mathrm{T}_\sigma(M_1), \mathrm{T}_{\mathrm{dR}}(M_1), \mathrm{T}_\ell(M_1), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ (for $i = 1, 2$) the system of realization defined by M_i for $i = 1, 2$. Consider an extension of $\mathrm{T}(M_1)$ by $\mathrm{T}(M_2)$ in the category $\mathcal{M}(k)$:

$$0 \longrightarrow \mathrm{T}(M_2) \xrightarrow{f} E \xrightarrow{g} \mathrm{T}(M_1) \longrightarrow 0$$

with $E = (E_\sigma, E_{\mathrm{dR}}, E_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$. In particular such an extension furnishes an extension in the Hodge realization, i.e. in the category $\mathcal{MHS}$ of mixed Hodge structures:

$$0 \longrightarrow \mathrm{T}_\sigma(M_2) \xrightarrow{f_\sigma} E_\sigma \xrightarrow{g_\sigma} \mathrm{T}_\sigma(M_1) \longrightarrow 0.$$

According to Proposition 3.1, modulo isogenies there exists a unique extension (M, i, j) of M_1 by M_2 which defines the extension E_σ. In other worlds in the category $\mathcal{MHS}$ we have an isomorphism

$$\epsilon : E_\sigma \longrightarrow \mathrm{T}_\sigma(M)$$

such that the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}_\sigma(M_2) \arrow[r, "f_\sigma"] \arrow[d, equal] & E_\sigma \arrow[r, "g_\sigma"] \arrow[d, "\epsilon"'] & \mathrm{T}_\sigma(M_1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{T}_\sigma(M_2) \arrow[r, "\mathrm{T}_\sigma(i)"] & \mathrm{T}_\sigma(M) \arrow[r, "\mathrm{T}_\sigma(j)"] & \mathrm{T}_\sigma(M_1) \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

where $\mathrm{T}_\sigma(i) : \mathrm{T}_\sigma(M_2) \to \mathrm{T}_\sigma(M)$ and $\mathrm{T}_\sigma(j) : \mathrm{T}_\sigma(M) \to \mathrm{T}_\sigma(M_1)$ are the morphisms in $\mathcal{MHS}$ induced by the morphisms of 1-motives $i : M_2 \to M$ and $j : M \to M_1$. The 1-motive M underlying the extension (M, i, j) is defined over $\mathbb{C}$. Let M_0 be a model of M over a finite extension k' of k. Since by [BLR90] 7.6 Proposition 5, the restriction of scalars $\mathrm{Res}_{k'/k} M_0$ is a 1-motive defined over k, we can assume that the 1-motive M is in fact defined over k. By Propositions 3.2 and 3.3, the extension (M, i, j) of M_1 by M_2 defines extensions also in the l-adic and in the de Rham realizations. The Hodge, the de Rham and the l-adic realizations of the data $M, i : M_2 \to M$ and $j : M \to M_1$ build the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}_\ell(M_2) \arrow[r, "\mathrm{T}_\ell(i)"] & \mathrm{T}_\ell(M) \arrow[r, "\mathrm{T}_\ell(j)"] & \mathrm{T}_\ell(M_1) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{T}_\sigma(M_2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \arrow[r, "\mathrm{T}_\sigma(i) \otimes \mathbb{Q}_\ell"] \arrow[u, "I_{\overline{\sigma},\ell}"] & \mathrm{T}_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \arrow[r, "\mathrm{T}_\sigma(j) \otimes \mathbb{Q}_\ell"] \arrow[u, "I_{\overline{\sigma},\ell}"] & \mathrm{T}_\sigma(M_1) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \arrow[r] \arrow[u, "I_{\overline{\sigma},\ell}"] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}_\sigma(M_2)\otimes_{\mathbb{Q}}\mathbb{C} \arrow[r, "\mathrm{T}_\sigma(i)\otimes\mathbb{C}"] \arrow[d, "I_{\sigma,\mathrm{dR}}"'] & \mathrm{T}_\sigma(M)\otimes_{\mathbb{Q}}\mathbb{C} \arrow[r, "\mathrm{T}_\sigma(j)\otimes\mathbb{C}"] \arrow[d, "I_{\sigma,\mathrm{dR}}"'] & \mathrm{T}_\sigma(M_1)\otimes_{\mathbb{Q}}\mathbb{C} \arrow[r] \arrow[d, "I_{\sigma,\mathrm{dR}}"'] & 0 \\
0 \arrow[r] & \mathrm{T}_{\mathrm{dR}}(M_2)\otimes_k\mathbb{C} \arrow[r, "\mathrm{T}_{\mathrm{dR}}(i)\otimes\mathbb{C}"] & \mathrm{T}_{\mathrm{dR}}(M)\otimes_k\mathbb{C} \arrow[r, "\mathrm{T}_{\mathrm{dR}}(j)\otimes\mathbb{C}"] & \mathrm{T}_{\mathrm{dR}}(M_1)\otimes_k\mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

We get therefore that the system of mixed realizations $\mathrm{T}(M) = (\mathrm{T}_\sigma(M), \mathrm{T}_{\mathrm{dR}}(M), \mathrm{T}_\ell(M), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ defined by M is an extension of $\mathrm{T}(M_1)$ by $\mathrm{T}(M_2)$ in the category $\mathcal{M}(k)$. Because of the comparison isomorphisms and of the commutativity of diagram (4.1), the isomorphism $\epsilon : E_\sigma \to \mathrm{T}_\sigma(M)$ implies the commutativity of the following diagram for the ℓ-adic realizations

$$\begin{tikzcd}
0 \arrow[d] & & & & 0 \arrow[d] \\
\mathrm{T}_\ell(M_2) \arrow[rrrr, equal] \arrow[dd, "f_\ell"'] & & & & \mathrm{T}_\ell(M_2) \arrow[dd, "\mathrm{T}_\ell(i)"] \\
& & \mathrm{T}_\sigma(M_2)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[ull, "I_{\overline{\sigma},\ell}"'] \arrow[urr, "I_{\overline{\sigma},\ell}"] \arrow[dl, "f_\sigma\otimes\mathbb{Q}_\ell"'] \arrow[dr, "\mathrm{T}_\sigma(i)\otimes\mathbb{Q}_\ell"] & & \\
E_\ell \arrow[dd, "g_\ell"'] & E_\sigma\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[l, "I_{\overline{\sigma},\ell}"'] \arrow[rr, "\epsilon\otimes\mathbb{Q}_\ell"] \arrow[dr, "g_\sigma\otimes\mathbb{Q}_\ell"'] & & \mathrm{T}_\sigma(M)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[r, "I_{\overline{\sigma},\ell}"] \arrow[dl, "\mathrm{T}_\sigma(j)\otimes\mathbb{Q}_\ell"] & \mathrm{T}_\ell(M) \arrow[dd, "\mathrm{T}_\ell(j)"] \\
& & \mathrm{T}_\sigma(M_1)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[dll, "I_{\overline{\sigma},\ell}"'] \arrow[drr, "I_{\overline{\sigma},\ell}"] & & \\
\mathrm{T}_\ell(M_1) \arrow[rrrr, equal] \arrow[d] & & & & \mathrm{T}_\ell(M_1) \arrow[d] \\
0 & & & & 0
\end{tikzcd}$$

The reader can check that we have an analogous commutative diagram also for the de Rham realizations. The commutativity of these diagrams (together with the commutativity of diagram (4.1)) means that the system of realizations E and $\mathrm{T}(M)$ are isomorphic as extensions of $\mathrm{T}(M_1)$ by $\mathrm{T}(M_2)$. Therefore we have proved that any extension of $\mathrm{T}(M_1)$ by $\mathrm{T}(M_2)$ in the category $\mathcal{M}(k)$ is defined by a unique 1-motive M modulo isogenies. $\square$

Remark 4.1. The hypothesis *"coming from geometry"* in Deligne's conjecture is essential, because in the category $\mathcal{MR}(k)$ of mixed realizations there are too many extensions. In order to explain this fact, we construct an extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(\mathbb{G}_m)$ in the category $\mathcal{MR}(k)$ which doesn't come from geometry. We start considering the 1-motive $M = [\mathbb{Z} \stackrel{u}{\to} \mathbb{G}_m], u(1) = 2$, defined over $\mathbb{Q}$, which is an extension of $\mathbb{Z}$ by $\mathbb{G}_m$. The mixed realization $\mathrm{T}(M)$ is the extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(\mathbb{G}_m)$ in the category of motives parametrized by the point 2 of $\mathbb{G}_m(\mathbb{Q})$, i.e. through the bijection

$$\mathbb{G}_m(\mathbb{Q}) \cong \mathrm{Ext}^1(\mathrm{T}(\mathbb{Z}), \mathrm{T}(\mathbb{G}_m))$$

the extension $\mathrm{T}(M)$ corresponds to the point 2 of $\mathbb{G}_m(\mathbb{Q})$. Denote by $E = (E_{\mathrm{H}}, E_{\mathrm{dR}}, E_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ the following mixed realization over $\mathrm{Spec}(\mathbb{Q})$:

- $E_{\mathrm{dR}} = \mathrm{T}_{\mathrm{dR}}(M)$. In particular, $E_{\mathrm{dR}} = \mathbb{Q} \oplus \mathbb{Q}$ is the trivial extension of $\mathbb{Q}$ by $\mathbb{Q}$;
- $E_{\mathrm{H}} = \mathrm{T}_{\mathrm{H}}(M)$. In particular, the lattice $E_{\mathbb{Z}}$ underlying E_{H} is generated by $(\log(2), 1), (2\pi i, 0)$ and it is a non trivial extension of $\mathbb{Z}$ by $\mathbb{Q}(1)$
- $E_\ell = \mathbb{Z}_\ell(1) \oplus \mathbb{Z}_\ell$ is the trivial extension of $\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell(1)$ for the Galois action $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- $I_{\mathrm{H,dR}} : E_{\mathrm{H}} \otimes_{\mathbb{Q}} \mathbb{C} \cong E_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ is the comparison isomorphism underlying the mixed realization $\mathrm{T}(M)$;
- $I_{\mathrm{H},\ell} : E_{\mathrm{H}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong E_\ell$ is the comparison isomorphism defined sending $(\log(2), 1)$ to $1 \in \mathbb{Z}_\ell$ and $(2\pi i, 0)$ to $\exp(\frac{2\pi i}{l^n}) \in \mathbb{Z}_\ell(1)$.

This mixed realization E is an extension of $\mathrm{T}(\mathbb{Z})$ by $\mathrm{T}(\mathbb{G}_m)$ in the category $\mathcal{MR}(\mathbb{Q})$ which isn't defined by a 1-motive extension of $\mathbb{Z}$ by $\mathbb{G}_m$.

References

[BK07] L. Barbieri-Viale and B. Kahn, *On the derived category of 1-motives I*, arXiv:0706.1498v1 [math.AG], 2007.

[Bl87] A. Beilinson, *Height pairing between algebraic cycles*, Contemp. Math., vol. 67, 1987, pp. 1–24.

[Be10] C. Bertolin, *Extensions of Picard stacks and their homological interpretation*, arXiv:1003.1866v1 [math.AG], 2010.

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 21. Springer-Verlag, Berlin, 1990.

[By83] J.-L. Brylinski, *1-motifs et formes automorphes (théorie arithmétique des domaines des Siegel)*, Pub. Math. Univ. Paris VII 15, 1983.

[D73] P. Deligne, *La formule de dualité globale*, Théorie des topos et cohomologie étale des schémas, Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973, pp. 481-587.

[D74] P. Deligne, *Théorie de Hodge III*, Inst. Hautes Études Sci. Publ. Math. No. 44, 1974, pp. 5–77.

[D89] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989, pp. 79–297.

[G71] J. Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.

Dip. di Matematica, Università di Milano, Via C. Saldini 50, I-20133 Milano

E-mail address: cristiana.bertolin@googlemail.com